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Strongly Unique Best Approximation in Banach Spaces, II

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1. INTRODUCTION AND PRELIMINARIES

Let X be a Banach space, and let M be a nonempty proper subset of X. Then an element $m \in M$ is called a *best approximation* in M to an element $x \in X$ if

$$\|x - m\| \leqslant \|x - y\| \tag{1.1}$$

for all y in M. If the set of all such elements m is nonempty then it is denoted by $\mathscr{P}_M(x)$. The mapping $\mathscr{P}_M : x \to \mathscr{P}_M(x)$ of X into 2^M is called a *metric projection.* Denote the domain of $\mathscr{P}_M$ by $\mathfrak{D}(\mathscr{P}_M)$. Clearly, we have $\mathfrak{D}(\mathscr{P}_M) \supset M$. Following [17], an element $m \in M$ is said to be a *strongly unique best approximation* in M to an element $x \in X$ if there exist a constant $c = c(x) > 0$ and an increasing continuous function φ: $[0, \infty) = \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(0) = 0$, such that the inequality

$$\varphi(\|x - m\|) \leqslant \varphi(\|x - y\|) - c\varphi(\|m - y\|) \tag{1.2}$$

holds for all y in M. Clearly, the strongly unique best approximation m is the unique best approximation in M to the element x, i.e., $\mathscr{P}_M(x) = \{m\}$.

It is now well known that the theory of best approximation can not be rich one without any additional assumptions about the set M. Therefore, several restrictions have been imposed on M in papers on nonlinear approximation theory. It seems that the most fruitful one is the concept of sun introduced by Efimov and Steckin [7]. We recall that M is said to be a *sun* if

$$m \in \mathscr{P}_M(x) \quad \text{implies} \quad m \in \mathscr{P}_M(m + \alpha(x - m)) \quad \text{for every} \quad \alpha > 0.$$

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One can easily show that M is a sun if and only if this implication is true only for $\alpha = 2$. Thus by (1.1), a set M is a sun if and only if the inequalities

$$\|x - m\| \leqslant \|x - (m + y)/2\|; \qquad y \in M, \tag{1.3}$$

hold for each $x \in \mathfrak{D}(\mathscr{P}_M)$ and $m \in \mathscr{P}_M(x)$. Clearly, by (1.3) it follows that every convex set is a sun. Note that if X is a strictly convex space, then every sun is a Chebyshev set, i.e., the set $\mathscr{P}_M(x)$ is one-element for each $x \in \mathfrak{D}(\mathscr{P}_M)$. Indeed, suppose that $x \in \mathfrak{D}(\mathscr{P}_M) \backslash M$, $m_1 \in \mathscr{P}_M(x)$ and $m_1 \neq m$. Then setting $y = m_1$ into (1.3) and using the triangle inequality for the norm on the right-side of (1.3), we get

$$\|x - m\| = \|x - m_1\| = \|x - (m + m_1)/2\|.$$

This means that the points m, $m_1 \neq m$ and $(m + m_1)/2$ belong to the sphere $\{z \in X: \|x - z\| = \|x - m\|\}$, which is impossible in a strictly convex space X.

In this paper we continue the study of strongly unique best approximations initiated in the paper [17]. More precisely, in Section 2 we show that a best approximation by elements of a sun in a uniformly convex Banach space is strongly unique locally. The global analogies of this result are presented in Section 3 and 4. In these sections there are proposed two different methods of proving (global) strong uniqueness of best approximations. In particular, we apply them to derive strong uniqueness theorems for the Lebesgue, Hardy, and Sobolev spaces. These methods are also applied to prove strong uniqueness of best approximations in some other Banach spaces. Finally, in Section 5 we show that a metric projection satisfies a Lipschitz condition of order $\alpha < 1$ in the most uniformly convex function spaces occurring in approximation theory.

2. Local Strong Uniqueness

Throughout this section we assume that X is a uniformly convex space with $\dim(X) \geqslant 2$. Then it is well known that the *modulus of convexity* $\delta_X: [0, 2] \to [0, 1]$ of X defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|x + y\|/2: x,\ y \in X,\ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\} \tag{2.1}$$

is an increasing continuous function. Moreover, we have $\delta_X(0) = 0$, $\delta_X(2) = 1$ and

$$\|(x + y)/2\| \leqslant r[1 - \delta_X(\|x - y\|/r)]; \qquad r > 0, \tag{2.2}$$

for all $x, y \in X$ such that $\|x\| \leqslant \|y\| \leqslant r$. Following Figiel [9], we denote by $\tilde{\delta}_X$ the maximal convex function majorated by δ_X. Clearly, the function $\tilde{\delta}_X$ is continuous on the open interval (0, 2) (see, e.g., [3, p. 26]) and

$$\tilde{\delta}_X(2) = \lim_{\varepsilon \to 2} \tilde{\delta}_X(\varepsilon).$$

By [13, Proposition 1.e.6 and Lemma 1.e.7] the function $\tilde{\delta}_X$ satisfies the estimates

$$d\delta(\varepsilon/2) \leqslant \tilde{\delta}_X(\varepsilon) \leqslant \delta_X(\varepsilon); \qquad 0 < \varepsilon < 2, \tag{2.3}$$

where d is a positive constant independent of ε. Hence it follows that $\tilde{\delta}_X(0) = 0$ and that $\tilde{\delta}_X$ is an increasing convex continuous function on $[0, 2]$.

THEOREM 2.1. *Let $m \in M$ be a best approximation in a sun $M \subset X$ to an element $x \in \mathfrak{D}(\mathscr{P}_M)$. Then there exist a constant $c = c(x, r) > 0$ and a continuously differentiable increasing convex function $\varphi = \varphi_r$: $[0, 2r] \to \mathbb{R}_+$, $\varphi(0) = 0$, such that the inequality*

$$\varphi(\|x - m\|) \leqslant \varphi(\|x - y\|) - c\varphi(\|m - y\|) \tag{2.4}$$

holds for all y in the ball

$$B_M(x, r) = \{y \in M: \|x - y\| \leqslant r\} \tag{2.5}$$

where $r \geqslant \operatorname{dist}(x, M) = \|x - m\|$ is an arbitrary fixed real number.

Proof. If $x \in M$, then $m = x$. Consequently, inequality (2.4) is true for a function φ and a constant $c \leqslant 1$. Therefore, we may suppose that $x \in \mathfrak{D}(\mathscr{P}_M) \setminus M$. Define the function φ: $[0, 2r] \to \mathbb{R}_+$ by $\varphi(0) = 0$ and

$$\varphi(t) = \varphi_r(t) := \int_0^t \frac{g(s)}{s}\, ds, \qquad 0 < t \leqslant 2r,$$

where $g(s) = \tilde{\delta}_X(s/r)$. Since $\tilde{\delta}_X(s/r)$, $0 \leqslant s \leqslant 2r$, is an increasing convex function, we have

$$(g(s_1) - g(s_0))/(s_1 - s_0) < (g(s_2) - g(s_0))/(s_2 - s_0)$$

for $0 \leqslant s_0 < s_1 < s_2 \leqslant 2r$ (cf. [12, p. 125]). If we put $s_0 = 0$ and use the Nördlander inequality [13, p. 63] for the modulus of convexity of X, then we obtain

$$\begin{aligned} 0 < g(s_1)/s_1 &< g(s_2)/s_2 \leqslant \delta_X(s_2/r)/s_2 \\ &\leqslant [1 - (1 - (s_2/r)^2/4)^{1/2}]/s_2 \leqslant (s_2/r)^2/(4s_2) \end{aligned}$$

for $0<s_1<s_2\leqslant 2r$. Thus the function $g(s)/s$, $0<s\leqslant 2r$, is an increasing continuous function and $g(s)/s\to 0$ as $s\to 0+$. Hence φ is a continuously differentiable increasing convex function on $[0,2r]$. Moreover, by the definition of φ we have

$$\varphi(\alpha t)=\alpha\int_0^t\frac{g(\alpha s)}{\alpha s}\,ds\leqslant\alpha\int_0^t\frac{g(s)}{s}\,ds=\alpha\varphi(t)\tag{2.6}$$

for every $\alpha\in(0,1]$, and

$$\varphi(t)\leqslant\int_0^t\frac{g(t)}{t}\,ds=\tilde{\delta}_X(t/r).\tag{2.7}$$

Now, if $m\in M$ is a best approximation in a sun to an element $x\in\mathfrak{D}(\mathscr{P}_M)\setminus M$ and $y\in B_M(x,r)$, then

$$0<\|x-m\|\leqslant\|x-y\|\leqslant r.$$

Hence using (1.3), (2.2)–(2.3) and (2.6)–(2.7) we derive

$$\begin{aligned}\varphi(\|x-m\|)&\leqslant\varphi(\|((x-m)+(x-y))/2\|)\\&\leqslant\varphi(\|x-y\|\,[1-\delta_X(\|m-y\|/\|x-y\|)])\\&\leqslant[1-\tilde{\delta}_X(\|m-y\|/\|x-y\|)]\,\varphi(\|x-y\|)\\&\leqslant\varphi(\|x-y\|)-\varphi(\|x-m\|)\,\tilde{\delta}_X(\|m-y\|/r)\\&\leqslant\varphi(\|x-y\|)-\varphi(\|x-m\|)\,\varphi(\|m-y\|)\\&=\varphi(\|x-y\|)-c\varphi(\|m-y\|),\end{aligned}$$

where

$$c=\varphi(\operatorname{dist}(x,M))>0.\tag{2.8}$$

This completes the proof. ∎

Let us remark that a best approximation in a set $M\subset X$ to an element $x\in\mathfrak{D}(\mathscr{P}_M)$ is a best approximation in a ball $B_M(x,r)$ to the element x for each $r\geqslant\operatorname{dist}(x,M)$. Thus Theorem 2.1 says that a best approximation in a sun of a uniformly convex space X to an element $x\in\mathfrak{D}(\mathscr{P}_M)$ is a strongly unique best approximation in a ball $B_M(x,r)$, $r\geqslant\operatorname{dist}(x,M)$, to the element x. It should be noticed that in the particular case, when M is a closed convex subset of a uniformly convex Banach space X, we have $\mathfrak{D}(\mathscr{P}_M)=X$ (see [3, p. 22]). We would like to emphasize that inequality (2.4) implies directly that $\mathscr{P}_M(x)$ is an one-element set for each $x\in\mathfrak{D}(\mathscr{P}_M)$. Indeed, if $m,m_1\in\mathscr{P}_M(x)$ then setting $y=m_1$ into (2.4) we obtain $c\varphi(\|m-m_1\|)\leqslant 0$, which is possible only when $m = m_1$. In the following we denote by $B(M, R)$ the ball centered at M of radius $R > 0$, i.e.,

$$B(M, R) = \{x \in X: \operatorname{dist}(x, M) \leqslant R\}.$$

COROLLARY 2.1. *Let $m \in M$ be a best approximation in a bounded sun $M \subset X$ to an element $x \in \mathfrak{D}(\mathscr{P}_M) \cap B(M, R)$, $R > 0$. Then there exist a constant $c = c(x) > 0$ and a continuously differentiable increasing function φ, $\varphi(0) = 0$, such that*

$$\varphi(\|x - m\|) \leqslant \varphi(\|x - y\|) - c\varphi(\|m - y\|)$$

for all y in M.

Proof. Let $r = R + \operatorname{diam}(M)$, where

$$\operatorname{diam}(M) = \sup\{\|y_1 - y_2\|: y_1, y_2 \in M\}.$$

Define the function $\varphi = \varphi_r$ as in the proof of Theorem 2.1. Since

$$0 \leqslant \|x - m\| \leqslant \|x - y\| \leqslant r; \qquad m = \mathscr{P}_M(x),$$

for all $x \in \mathfrak{D}(\mathscr{P}_M) \cap B(M, R)$ and $y \in M$, we can repeat mutatis mutandis the proof of Theorem 2.1 in order to prove the corollary. ∎

3. Global Strong Uniqueness

In this section we improve Theorem 2.1 for some class of uniformly convex spaces X. For this purpose, we assume throughout the section that $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing convex continuous function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. We shall say that a uniformly convex space X has *modulus of convexity of the type* φ if there is a constant K, $0 < K < \infty$, such that

$$\delta_X(\varepsilon) \geqslant K\varphi(\varepsilon), \qquad 0 \leqslant \varepsilon \leqslant 2. \tag{3.1}$$

The function φ is said to be *submultiplicative* if there is a constant L, $0 < L < \infty$, such that the inequality

$$\varphi(ts) \leqslant L\varphi(t)\,\varphi(s) \tag{3.2}$$

holds for all positive t and s. It should be noticed that every uniformly convex space X has modulus of convexity of the type φ provided that the increasing convex continuous function φ is defined by $\varphi(t) = d\sigma(t)$ with

$$\sigma(t) = \int_0^t \frac{\tilde{\delta}_X(s)}{s}\, ds, \qquad t > 0,$$

where $d = 1/\sigma(1)$ and $\tilde{\delta}_X(s)$, $s \geqslant 0$, is an increasing extension to $\mathbb{R}_+$ of the maximal convex function $\tilde{\delta}_X(s)$, $0 \leqslant s \leqslant 2$, majorated by $\delta_X(s)$, $0 \leqslant s \leqslant 2$. In this case we have $K = 1/d = \sigma(1)$.

THEOREM 3.1. *Let M be a sun in a uniformly convex space X having modulus of convexity δ_X of the type φ. Assume that φ is a submultiplicative function and that $m \in M$ is a best approximation in M to an element $x \in \mathfrak{D}(\mathscr{P}_M)$. Then the inequality*

$$\varphi(\|x-m\|) \leqslant \varphi(\|x-y\|) - KL^{-1}\varphi(\|m-y\|) \tag{3.3}$$

holds for all y in M, where K and L are as in (3.1)–(3.2).

Proof. Since $\varphi(1) = 1$, it follows from (3.1)–(3.2) that $K \leqslant 1$ and $L \geqslant 1$. Therefore, without loss of generality, we may suppose that $x \neq m$, i.e.,

$$0 < \|x-m\| \leqslant \|x-y\|, \qquad y \in M. \tag{3.4}$$

Since $\varphi(0) = 0$ and φ is a convex function, we have

$$\varphi(ts) = \varphi(ts + (1-t)\cdot 0) \leqslant t\varphi(s)$$

for all $0 \leqslant t \leqslant 1$ and $s \in \mathbb{R}_+$. This in conjunction with (1.3), (2.2), (3.1)–(3.2), (3.4) and the fact that φ is an increasing function gives

$$\begin{aligned}
\varphi(\|x-m\|) &\leqslant \varphi(\|((x-m)+(x-y))/2\|) \\
&\leqslant \varphi(\|x-y\|\,[1-\delta_X(\|m-y\|/\|x-y\|)]) \\
&\leqslant [1-\delta_X(\|m-y\|/\|x-y\|)]\;\varphi(\|x-y\|) \\
&\leqslant \varphi(\|x-y\|) - K\varphi(\|m-y\|/\|x-y\|)\,\varphi(\|x-y\|) \\
&\leqslant \varphi(x-y\|) - KL^{-1}\varphi(\|m-y\|)
\end{aligned}$$

for all $y \in M$. This completes the proof. ∎

Remark 3.1. In a recent paper Prus and Smarzewski [15] established Theorem 3.1 for the function $\varphi(t) = t^q$, $q \geqslant 2$, but with the constant $c = KL^{-1}$ replaced by an unknown constant.

The theorem says that the element $m \in M$ is a strongly unique best approximation to the element $x \in \mathfrak{D}(\mathscr{P}_M)$ with the constant $c = KL^{-1}$ independent of x. It can be applied to the most interesting uniformly convex spaces occurring in approximation theory. For example, let

$X = L_p = L_p(S, \Sigma, \mu)$, $1 < p < \infty$, be the Banach space of all μ-measurable extended scalar valued functions (equivalence classes) x on S such that

$$\|x\| = \|x\|_p := \left[\int_S |x(s)|^p \, \mu(ds)\right]^{1/p} < \infty$$

where (S, Σ, μ) denotes a positive measure space. Then we have

COROLLARY 3.1. *Let M be a sun in L_p, $1 < p < \infty$. If $m \in M$ is a best approximation in M to an element $x \in \mathfrak{D}(\mathscr{P}_M)$ then*

$$\|x - m\|^q \leqslant \|x - y\|^q - c_p \|m - y\|^q \tag{3.5}$$

for all y in M, where $q = \max(2, p)$ and

$$c_p = \begin{cases} (p-1)/8, & \text{if } 1 < p \leqslant 2, \\ 1/(p2^p) & \text{if } 2 \leqslant p < \infty. \end{cases} \tag{3.6}$$

Proof. First, we note that

$$\delta_{L_p}(\varepsilon) \geqslant c_p \varepsilon^q, \qquad 0 \leqslant \varepsilon \leqslant 2. \tag{3.7}$$

Indeed, if $p \geqslant 2$ then this inequality can be easily deduced from the formula for δ_{L_p} given in [10] (cf. also [8, p. 300]). Further, if $1 < p \leqslant 2$ then inequality (3.7) can be found in [14]. Now, let us set $\varphi(t) = t^q$, $t \geqslant 0$. This function satisfies all assumptions of Theorem 3.1 with $K = c_p$ and $L = 1$. Thus by applying Theorem 3.1 we obtain the desired result. ∎

The corollary has been proved recently in [18] ([15]) for a closed convex subset of L_p with $2 \leqslant p < \infty$ ($1 < p \leqslant 2$, respectively). The constants c_p given in [18] ([15]) are greater (smaller, respectively) than the constant c_p defined by (3.6). The same result can be established for the Banach space $X = H^p$, $1 < p < \infty$, of all functions x analytic in the unit disc $|z| < 1$ of the complex plane and such that

$$\|x\| = \|x\|_p := \lim_{r \to 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |x(re^{i\theta})|^p \, d\theta\right)^{1/p} < \infty.$$

COROLLARY 3.2. *Let M be a sun in H^p, $1 < p < \infty$. If $m \in M$ is a best approximation in M to an element $x \in \mathfrak{D}(\mathscr{P}_M)$ then*

$$\|x - m\|^q \leqslant \|x - y\|^q - c_p \|m - y\|^q$$

for all y in M, where $q = \max(2, p)$ and c_p is as in (3.6).

Proof. Let $L_p = L_p(S, \Sigma, \mu)$, where μ is the measure of Lebesgue in the interval $S = (0, 2\pi)$ such that $\mu(S) = 1$. Denote by $\hat{f}$ the boundary function in L_p corresponding to a function f in H^p, i.e., let $\hat{f}(\theta)$ be the L_p-limit of $f(re^{i\theta})$ as $r \to 1-$ [6, p. 21]. The mapping $\mathscr{F}: f \to \hat{f}$ of H^p into L_p is an isometric isomorphism [6, p. 35]. Therefore, we have

$$\delta_{H^p}(\varepsilon) \geqslant \delta_{L_p}(\varepsilon) \geqslant c_p \varepsilon^q, \qquad 0 \leqslant \varepsilon \leqslant 2.$$

Finally, one can apply Theorem 3.1 in order to finish the proof. ∎

Finally, we note that a power function is also admissible in Theorem 3.1 when the Banach space X is p-convex and q-concave [9]. More precisely, we have

COROLLARY 3.4. *Suppose that the Banach space is p-convex and s-concave, where* $1 < p \leqslant s < \infty$. *Let* $m \in M$ *be a best approximation in a sun* $M \subset X$ *to an element* $x \in \mathfrak{D}(\mathscr{P}_M)$. *Then*

$$\|x - m\|^q \leqslant \|x - y\|^q - c\,\|m - y\|^q$$

for all y in M, where $q = \max(2, s)$ *and*

$$c = q^{-1}(\max(2, 2/(p-1)^{1/2}))^{-q}.$$

Proof. By Proposition 24 of Figiel [9] we have

$$\delta_X(\varepsilon) \geqslant c\varepsilon^q, \qquad 0 \leqslant \varepsilon \leqslant 2,$$

which in view of Theorem 3.1 completes the proof. ∎

The corollary in conjunction with the Proposition 1 of Figiel [8] can be used to prove strong uniqueness in Sobolev spaces. We do not present details here, because this method is less elementary than the method proposed in the next section. Furthermore, the constants given in the next section are much better than the constants which would be given here.

4. Another Approach to Global Strong Uniqueness

In this section we consider a new method of proving global strong uniqueness of best approximations. The method does not use the notion of modulus of convexity.

THEOREM 4.1. *Suppose that there exist a positive constant K and an increasing continuous function* $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(0) = 0$, *such that the inequality*

$$\varphi\left(\left\|\frac{u+v}{2}\right\|\right) \leqslant \tfrac{1}{2}[\varphi(\|u\|) + \varphi(\|v\|)] - K\varphi(\|u - v\|) \tag{4.1}$$

holds for all u, v in X. Let $m \in M$ be a best approximation in a sun $M \subset X$ to an element $x \in \mathfrak{D}(\mathscr{P}_M)$. Then

$$\varphi(\|x-m\|) \leqslant \varphi(\|x-y\|) - 2K\varphi(\|m-y\|) \tag{4.2}$$

for all y in M.

Proof. By (1.3) we have

$$\varphi(\|x-m\|) \leqslant \varphi(\|((x-m)+(x-y))/2\|)$$

for all y in M. Hence by using (4.1) we get

$$\varphi(\|x-m\|) \leqslant \tfrac{1}{2}(\varphi(\|x-m\|) + \varphi(\|x-y\|)) - K\varphi(\|m-y\|),$$

which is equivalent to (4.2). ∎

The theorem can be easily applied to prove strong uniqueness of best approximations in L_p and H^p spaces.

COROLLARY 4.1. *Let M be a sun in the space X, where $X = L_p$ or $X = H^p$ and $1 < p < \infty$. If $m \in M$ is a best approximation in M to an element $x \in \mathfrak{D}(\mathscr{P}_M)$ then*

$$\|x-m\|^q \leqslant \|x-y\|^q - c_p\|m-y\|^q \tag{4.3}$$

for all y in M, where $q = \max(2, p)$ and

$$c_p = \begin{cases} p(p-1)/4 & \text{if } \ 1 < p \leqslant 2, \\ 2^{1-p} & \text{if } \ 2 \leqslant p < \infty. \end{cases} \tag{4.4}$$

Proof. We recall the Clarkson inequality [4, Theorem 2],

$$\|u+v\|^p + \|u-v\|^p \leqslant 2^{p-1}(\|u\|^p + \|v\|^p), \tag{4.5}$$

which holds for all $u, v \in L_p$, $p \geqslant 2$. It is clear that this inequality is equivalent to inequality (4.1) with $\varphi(t) = t^p$ and $K = 2^{-p}$, which in view of Theorem 4.1 completes the proof in the case $p \geqslant 2$. Further, by the Meir inequality [14, Inequality (2.3)] we have

$$\left\|\frac{u+v}{2}\right\|^2 \leqslant \frac{1}{2}(\|u\|^2 + \|v\|^2) - \frac{p(p-1)}{8}\|u-v\|^2$$

for all $u, v \in L_p$, $1 < p \leqslant 2$. Hence we can apply Theorem 4.1 in order to finish the proof for L_p spaces. Since the space H^p is isometrically isomorphic with a subspace of the Lebesgue space $L_p(0, 2\pi)$ (cf. the proof of Corollary 3.2), we readily conclude that the corollary is also true for H^p spaces. ∎

We remark that the constant c_p is now better than the constant given in Corollaries 3.1 and 3.2. Now we prove an auxiliary lemma which will be needed below.

LEMMA 4.1. *The inequality*

$$(t^p + s^p)\left(\frac{t+s}{2}\right)^{2-p} \leqslant t^2 + s^2, \qquad 1 \leqslant p \leqslant 2, \tag{4.6}$$

holds for all nonnegative numbers t and s.

Proof. Inequality (4.6) is obvious when $t = s$, $s = 0$, or $t = 0$. Therefore, without loss of generality, we may suppose that $0 < t < s$. Dividing both sides of inequality (4.6) by s^2, we get the equivalent inequality

$$f(p) := z^2 + 1 - (z^p + 1)\left(\frac{z+1}{2}\right)^{2-p} \geqslant 0, \qquad 1 \leqslant p \leqslant 2,$$

where $z = t/s$ is an arbitrary fixed number in the interval $(0, 1)$. We note that

$$h(p) := \left(\frac{z+1}{2}\right)^{p-2} f'(p) = (z^p + 1)\ln\frac{z+1}{2} - z^p \ln z.$$

Since

$$h'(p) = z^p\left(\ln\frac{z+1}{2} - \ln z\right)\ln z \leqslant 0$$

and $h(1) \leqslant 0$, it follows that $f'(p) \leqslant 0$ for $1 \leqslant p \leqslant 2$. This in conjunction with the fact that $f(2) = 0$ implies that $f(p) \geqslant 0$ for $1 \leqslant p \leqslant 2$, which finishes the proof. ∎

Now, let Ω be an open subset of R^n. Denote by $H^{k,p} = H^{k,p}(\Omega)$; $k \geqslant 0$ and $1 < p < \infty$, the Sobolev space [1, p. 149] of distributions x such that $D^\alpha x \in L_p(\Omega)$ for all $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant k$. We recall that the norm in $H^{k,p}$ is defined by

$$\|x\| = \|x\|_{k,p} := \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha x(\omega)|^p \, d\omega\right)^{1/p}.$$

COROLLARY 4.2. *Let M be a sun in $H^{k,p}$, where $k \geqslant 0$ and $1 < p < \infty$. If $m \in M$ is a best approximation in M to an element $x \in \mathfrak{D}(\mathscr{P}_M)$ then*

$$\|x - m\|^q \leqslant \|x - y\|^q - c_p \|m - y\|^q \tag{4.7}$$

for all y in M, where $q = \max(2, p)$ and c_p is as in (3.6).

Proof. First we consider the case $p \geqslant 2$. Since $D^{\alpha}x \in L_p(\Omega)$ for every $x \in H^{k,p}$, we can use inequality (4.5) to get

$$\left\| \frac{D^{\alpha}(u+v)}{2} \right\|_p^p \leqslant \frac{1}{2} (\|D^{\alpha}u\|_p^p + \|D^{\alpha}v\|_p^p) - \frac{1}{2} c_p \|D^{\alpha}(u-v)\|_p^p \tag{4.8}$$

for all u, v in $H^{k,p}$, where α is a multiindex such that $|\alpha| \leqslant k$ and $\|\cdot\|_p$ denotes the norm in $L_p(\Omega)$. If we sum up inequalities (4.8) over $|\alpha| \leqslant k$, then we get

$$\left\| \frac{u+v}{2} \right\|^p \leqslant \frac{1}{2} (\|u\|^p + \|u\|^p) - \frac{1}{2} c_p \|u-v\|^p$$

for all u, v in $H^{k,p}$. Hence by applying Theorem 4.1 we obtain inequality (4.7). Now suppose that $1 < p < 2$. Then by Theorem 1 of Meir [14], we derive

$$\begin{aligned} \left\| \frac{D^{\alpha}(u+v)}{2} \right\|_p^p \leqslant \frac{1}{2} (\|D^{\alpha}u\|_p^p + \|D^{\alpha}v\|_p^p) - \frac{1}{2} c_p \|D^{\alpha}(u-v)\|_p^2 \\ \times \left\| \frac{|D^{\alpha}u| + |D^{\alpha}v|}{2} \right\|_p^{p-2} \end{aligned} \tag{4.9}$$

for all $|\alpha| \leqslant k$ and u, v in $H^{k,p}$ such that $|D^{\alpha}u| + |D^{\alpha}v|$ is not equal to the null θ of $L_p(\Omega)$. We need the Radon inequality [11, Theorem 51],

$$\sum t_{\alpha}^{2/p} s_{\alpha}^{1-2/p} \geqslant \left(\sum t_{\alpha} \right)^{2/p} \left(\sum s_{\alpha} \right)^{1-2/p}; \qquad t_{\alpha} \geqslant 0,\ s_{\alpha} > 0.$$

Summing inequalities (4.9) over α's such that $|\alpha| \leqslant k$ and $|D^{\alpha}u| + |D^{\alpha}v| \neq \theta$, and using the Radon inequality, we obtain

$$\left\| \frac{u+v}{2} \right\|^p \leqslant \frac{1}{2} (\|u\|^p + \|v\|^p) - \frac{1}{2} c_p \|u-v\|^2 \left(\sum \left\| \frac{|D^{\alpha}u| + |D^{\alpha}v|}{2} \right\|_p^p \right)^{1-2/p}.$$

Hence by Minkowski's inequality we get

$$\left\| \frac{u+v}{2} \right\|^p \leqslant \frac{1}{2} (\|u\|^p + \|v\|^p) - \frac{1}{2} c_p \|u-v\|^2 \left(\frac{\|u\| + \|v\|}{2} \right)^{p-2}.$$

This in conjunction with Lemma 4.1 implies that

$$\left\|\frac{u+v}{2}\right\|^2 \leqslant \left\|\frac{u+v}{2}\right\|^p \left(\frac{\|u\|+\|v\|}{2}\right)^{2-p}$$
$$\leqslant \frac{1}{2}(\|u\|^p+\|v\|^p)\left(\frac{\|u\|+\|v\|}{2}\right)^{2-p}$$
$$-\frac{1}{2}c_p\|u-v\|^2 \leqslant \frac{1}{2}(\|u\|^2+\|v\|^2)-\frac{1}{2}c_p\|u-v\|^2$$

for all u, v in $H^{k,p}$. Hence by Theorem 4.1 the proof is completed. ▮

The proof of Corollary 4.2 can be easily extended to the spaces $l_q(L_p)$ [13, p. 46]. For this purpose, let $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $\alpha \in A$, be a sequence of positive measure spaces, where the index set A is finite or countable. Given a sequence of linear subspaces X_α in $L_p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, we denote by $L_{q,p}$; $1<p<\infty$ and $q=\max(2, p)$, the linear space of all sequences

$$x=\{x_\alpha \in X_\alpha : \alpha \in A\} \in l_q(A)$$

equipped with the norm

$$\|x\| = \|x\|_{p,q} := \left(\sum_{\alpha \in A} \|x_\alpha\|^q_{p,\alpha}\right)^{1/q},$$

where $\|\cdot\|_{p,\alpha}$ denotes the norm in $L_p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$.

COROLLARY 4.3. *Suppose that m is a best approximation in a sun $M \subset L_{q,p}$; $1<p<\infty$ and $q=\max(2, p)$, to an element $x \in \mathfrak{D}(\mathscr{P}_M)$. Then*

$$\|x-m\|^q \leqslant \|x-y\|^q - c_p\|m-y\|^q$$

for all y in M, where c_p is as in (3.6).

Proof. Replace the symbols $D^\alpha u$, $D^\alpha v$, $L_p(\Omega)$, $H^{k,p}$ and $\|\cdot\|_p$ occurring in the proof of Corollary 4.2 by $u_\alpha \in X_\alpha$, $v_\alpha \in X_\alpha$, $L_p(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)$, $L_{q,p}$ and $\|\cdot\|_{p,\alpha}$, respectively. Next, repeat mutatis mutandis this proof. ▮

Finally, let $L_p = L_p(S_1, \Sigma_1, \mu_1)$ and $L_q = L_q(S_2, \Sigma_2, \mu_2)$, where $1<p<\infty$, $q=\max(2, p)$ and (S_i, Σ_i, μ_i) are positive measure spaces. Denote by $L_q(L_p)$ the Banach space [5, III.2.10] of all measurable L_p-valued functions f on S_2 such that

$$\|x\| := \left(\int_{S_2} \|x(s)\|_p^q \, \mu_2(ds)\right)^{1/q}.$$

Then we have

COROLLARY 4.4. *Let $m \in M$ be a best approximation in a sun $M \subset L_q(L_p)$ to an element $x \in \mathfrak{D}(\mathscr{P}_M)$. Then*

$$\|x-m\|^q \leqslant \|x-y\|^q - c_p \|m-y\|^q$$

for all y in M, where $q = \max(2, p)$ and c_p is as in (3.6).

Proof. Let u and v be two elements of $L_q(L_p)$. Then $u(t)$ and $v(t)$ belong to L_p for each $t \in S_2$. By Theorem 4.1, it is sufficient to prove that

$$\left\|\frac{u+v}{2}\right\|^q \leqslant \frac{1}{2}(\|u\|^q + \|v\|^q) - \frac{1}{2} c_p \|u-v\|^q. \tag{4.10}$$

There is no loss of generality in assuming that

$$z(s) = \tfrac{1}{2}(|u(s)| + |v(s)|)$$

does not vanish (otherwise, we integrate only over $\{s \in S_2 : z(s) > 0\}$). If $1 < p \leqslant 2$, then we put $u(s)$ and $v(s)$ into the Meir inequality [14, Theorem 1],

$$\left\|\frac{u(s)+v(s)}{2}\right\|_p^p \leqslant \frac{1}{2}(\|u(s)\|_p^p + \|v(s)\|_p^p) - \frac{1}{2} c_p \|u(s)-v(s)\|_p^2 \|z(s)\|_p^{p-2}.$$

Integration of both sides and application of Hölder's inequality yields

$$\begin{aligned}\left\|\frac{u+v}{2}\right\|^p &\leqslant \frac{1}{2}(\|u\|^p + \|v\|^p) - \frac{1}{2} c_p \int_{S_2} (\|u(s)-v(s)\|_p^p \\ &\quad \times \|z(s)\|_p^{p(p-2)/2})^{2/p} \mu_2(ds) \leqslant \frac{1}{2}(\|u\|^p + \|v\|^p) \\ &\quad - \frac{1}{2} c_p \left(\int_{S_2} \|u(s)-v(s)\|_p^p \mu_2(ds)\right)^{2/p} \left(\int_{S_2} \|z(s)\|_p^p \mu_2(ds)\right)^{1-2/p} \\ &\leqslant \frac{1}{2}(\|u\|^p + \|v\|^p) - \frac{1}{2} c_p \|u-v\|^2 \left(\frac{\|u\| + \|v\|}{2}\right)^{p-2}.\end{aligned} \tag{4.11}$$

Hence by applying Lemma 4.1, we obtain inequality (4.10). Finally, if $p \geqslant 2$ then inequality (4.10) follows directly from the Clarkson inequality (4.5). ∎

We remark that inequalities (4.10), (4.11) can be used to prove the estimates for moduli of convexity of $L_q(L_p)$ spaces,

$$\delta_{L_q(L_p)}(\varepsilon) \geqslant d_p \varepsilon^q, \qquad 0 \leqslant \varepsilon \leqslant 2, \tag{4.12}$$

where $q = \max(2, p)$ and

$$d_p = \begin{cases} (p-1)/8, & \text{if} \quad 1 < p \leqslant 2, \\ 2^{-p}/p, & \text{if} \quad p \geqslant 2. \end{cases}$$

Indeed, applying inequalities (4.10)–(4.11), we get

$$\frac{1}{2} c_p \varepsilon^q \leqslant 1 - \left\| \frac{u+v}{2} \right\|^p \leqslant p \left(1 - \left\| \frac{u+v}{2} \right\| \right)$$

for all u, v in $L_q(L_p)$ such that $\|u\| = \|v\| = 1$ and $\|u - v\| = \varepsilon$. Hence we immediately obtain (4.12). Clearly, the same estimates can be similarly proved for moduli of convexity of spaces $H^{k,r}$ and $L_{q,p}$. One can notice that if $L_q = l_q^1$ then the estimate (4.12) for the modulus of convexity of $L_q(L_p) = L_p$, $1 < p \leqslant 2$, coincides with the estimate given recently by Meir [14, Corollary 1]. It should be also noticed that a super-reflexive space X can be renormed in such a way that Theorem 4.1 can be applied with $\varphi(t) = t^q$ for some $q \geqslant 2$. This follows directly from Theorems 18.2 and 18.7 presented in [16] (cf. also [19, Section III.2]).

5. Some Applications of Strong Uniqueness

Recently, we have proved in [17] that the metric projection $\mathscr{P}_M$ is locally Lipschitzian of order $1/p$ for a linear subspace M of L_p, $2 \leqslant p < \infty$. Moreover, $\mathscr{P}_M$ is also Lipschitzian of order $2/p$ which was proved by Björnestal in [2]. We are indebted to the referee for this reference. Now we can extend our result as follows.

Theorem 5.1. *Let M be a sun in X such that $0 \in M$. Suppose that there exist a positive constant $c \leqslant 1$ and $q \geqslant 2$ such that the inequality*

$$\|x - m\|^q \leqslant \|x - y\|^q - c\,\|m - y\|^q \tag{5.1}$$

holds for any $x \in \mathfrak{D}(\mathscr{P}_M)$, $m \in \mathscr{P}_M(x)$ and *$y \in M$. Then we have*

$$\|\mathscr{P}_M(x_1) - \mathscr{P}_M(x_2)\| \leqslant d r^{1-1/q} \|x_1 - x_2\|^{1/q} \tag{5.2}$$

for all x_1, x_2 in a ball $B(r) = \{x \in \mathfrak{D}(\mathscr{P}_M) \colon \|x\| \leqslant r\}$, where

$$d = (q/c)^{1/q} (1 + c^{-q})^{1-1/q} \leqslant 2 + c^{-q}.$$

Proof. Denote $m_1 = \mathscr{P}_M(x_1)$ and $m_2 = \mathscr{P}_M(x_2)$. Since $0 \in M$, we have

$$\|x_i - m_i\| \leqslant \|x_i\| \leqslant r; \qquad i = 1, 2.$$

Moreover, putting $y=0$ into (5.1) we conclude that

$$\|m_i\| \leqslant c^{-1/q}\|x_i\| \leqslant c^{-1/q}r; \qquad i=1, 2.$$

Hence by applying twice inequality (5.1) and using the inequality

$$|t^q - s^q| \leqslant qr^{q-1}|t-s|; \qquad 0 \leqslant t, s \leqslant r,$$

we obtain

$$\begin{aligned} c\,&\|m_1 - m_2\|^q \\ &\leqslant \tfrac{1}{2}(\|x_1 - m_2\|^q - \|x_2 - m_2\|^q) + \tfrac{1}{2}(\|x_2 - m_1\|^q - \|x_1 - m_1\|^q) \\ &\leqslant \tfrac{1}{2}q(r + rc^{-1/q})^{q-1}(|\|x_1 - m_2\| - \|x_2 - m_2\|| + |\|x_2 - m_1\| - \|x_1 - m_1\||) \\ &\leqslant q(r + rc^{-1/q})^{q-1}\|x_1 - x_2\|. \end{aligned}$$

This completes the proof of (5.2). Finally, the estimate for d follows from the well-known inequality between weighted arithmetic and geometric means [11, Inequality 2.5.2]. ∎

In particular, it follows from results presented in the proceeding two sections that assumptions of Theorem 5.1 are satisfied when M, $0 \in M$, is a sun in spaces L_p, H^p, $H^{k,p}$, $L_{q,p}$, and $L_q(L_p)$, where $1 < p < \infty$ and $q = \max(2, p)$. Additionally, by Corollary 3.1, this is also true when the space X is p-convex and s-concave, $1 < p \leqslant s < \infty$. In this case, $q = \max(2, s)$.

Strong uniqueness can be also applied to establish the rate of convergence of numerical algorithms for computing best approximations. For this purpose, let m be a best approximation in a sun $M \subset X$ to an element $x \in \mathfrak{D}(\mathscr{P}_M)$. Then

$$\lambda := \|x - m\|^q = \inf\{\|x - y\|^q \colon y \in M\}.$$

Suppose that $\{m_i\} \subset M$ is a minimizing sequence for the functional $f(y) = \|x - y\|^q$, $y \in M$, produced by a numerical algorithm, i.e., that

$$\lambda_i := \|x - m_i\|^q \to \lambda \qquad \text{as} \quad i \to \infty.$$

Then we have

THEOREM 5.2. *Under the assumptions of Theorem* 5.1, *the minimizing sequence* $\{m_i\}$ *converges to* m *with the rate*

$$\|m - m_i\|^q \leqslant (\lambda_i - \lambda)/c.$$

Proof. Replace y by m_i in inequality (5.1). ∎

Finally, we note that Theorems 5.1 and 5.2 can be extended to the case when

$$\varphi(\|x-m\|) \leqslant \varphi(\|x-y\|) - c\varphi(\|m-y\|)$$

for all y in M, where φ is an increasing convex continuous function on $\mathbb{R}_+$.

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